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Chebyshev Additive Weight Approximation by Maximal Families

RYSZARD SMARZEWSKI

Department of Numerical Methods, M. Curie–Skłodowska University, 20–031 Lublin, Poland

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1. INTRODUCTION

Let $C(X)$ be the space of real-valued functions defined and continuous on a compact Hausdorff space normed by

$$\|f\| = \max\{|f(x)|: x \in X\}.$$

For any $f \in C(X)$ and $\alpha \geqslant 0$ define the following closed sets:

$$Z_f = \{x \in X: f(x) = 0\},$$
$$M_f^+(\alpha) = \{x \in X: f(x) \geqslant \alpha\},$$
$$M_f^-(\alpha) = \{x \in X: f(x) \leqslant -\alpha\}$$

and

$$M_f(\alpha) = M_f^+(\alpha) \cup M_f^-(\alpha).$$

If $\alpha = \|f\|$, then we briefly denote the last three sets by M_f^+, M_f^- and M_f respectively. In the following we shall assume that G is a proper subset of $C(X)$, w is a fixed nonnegative function in $C(X)$ and $f \in C(X)\backslash G$.

DEFINITION 1. An element $g \in G$ is said to be a best additive weight approximation in G to f if

$$\|\,|f-g| + w\| \leqslant \|\,|f-h| + w\|$$

for all $h \in G$.

The main purpose of this paper is to obtain characterization theorems of Kolmogorov and alternation type for additive weight approximation by so-

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called maximal families. The precise definitions and results are given in the following sections. Observe that our paper is a continuation of the investigations carried out by M. J. Gillotte and H. W. McLaughlin in [3].

2. Theorems of Kolmogorov Type

THEOREM 1. *A sufficient condition for an element $g \in G$ to be a best additive approximation in G to f is that the inequality*

$$\max\{[g(x)-h(x)] \operatorname{sign}[f(x)-g(x)]: x \in M_{|f-g|+w}\} \geqslant 0 \tag{1}$$

be satisfied for all $h \in G$ or $Z_{f-g} \cap M_{|f-g|+w} \neq \varnothing$.

Proof. Firstly suppose that (1) holds for each $h \in G$ and

$$Z_{f-g} \cap M_{|f-g|+w} = \varnothing. \tag{2}$$

Note that the nonempty set $M_{|f-g|+w}$ is closed, so it is compact. Hence, from a continuity of $|f-g|$ and (2) we have

$$\delta = \min\{|f(x)-g(x)|: x \in M_{|f-g|+w}\} > 0.$$

This implies that the sets $U = M_{|f-g|+w} \cap M^{+}_{f-g}(\delta)$ and $V = M_{|f-g|+w} \cap M^{-}_{f-g}(\delta)$ are disjoint and closed.

Hence the function

$$[g(x)-h(x)] \operatorname{sign}[f(x)-g(x)] = \begin{cases} g(x)-h(x), & \text{if } x \in U, \\ h(x)-g(x), & \text{if } x \in V, \end{cases}$$

is continuous on the set $U \cup V = M_{|f-g|+w}$ and consequently achieves its maximum on this set, e.g., at the point z.

Now, from (1) we obtain

$$\begin{aligned} \||f-g|+w\| &\leqslant \||f-g|+w\| \\ &\quad + \max\{[g(x)-h(x)] \operatorname{sign}[f(x)-g(x)]: x \in M_{|f-g|+w}\} \\ &= |f(z)-g(z)| + [g(z)-h(z)] \operatorname{sign}[f(z)-g(z)] + w(z) \\ &= [f(z)-h(z)] \operatorname{sign}[f(z)-g(z)] + w(z) \leqslant \||f-h|+w\|, \end{aligned}$$

which implies that g is a best additive weight approximation to f in G. Secondly, if (2) is not true, i.e., if there exists $z \in Z_{f-g} \cap M_{|f-g|+w}$, then we have

$$\||f-g|+w\| = w(z) \leqslant |f(z)-h(z)| + w(z) \leqslant \||f-h|+w\|,$$

which also implies that g is a best additive weight approximation. ∎

Theorem 1 suggests the following problem: What are minimal assumptions regarding a subset G of $C(X)$ guaranteeing that condition (1) is also a necessary condition for g to be a best additive weight approximation? Now, we shall deal with this problem directly.

DEFINITION 2 (see [7, 8]). A subset G of $C(X)$ has the weak betweenness property if for any two distinct elements g and h in G and every nonempty closed subset D of X such that

$$\min\{|h(x)-g(x)|: x \in D\} > 0$$

there exists a sequence $\{g_i\}$ in G such that

(i) $\lim_{i \to \infty} \|g - g_i\| = 0,$

and

(ii) $\min\{[h(x)-g_i(x)][g_i(x)-g(x)]: x \in D\} > 0$

for all integers i.

THEOREM 2. *Let G have the weak betweenness property. Then a necessary condition for an element $g \in G$ to be a best additive weight approximation in G to $f \in C(X)$ is that inequality* (1) *be satisfied for all $h \in G$ or $Z_{f-g} \cap M_{|f-g|+w} \neq \varnothing$.*

Proof. Let us suppose, on the contrary, that there exists $h \in G$ such that inequality (1) does not hold and

$$Z_{f-g} \cap M_{|f-g|+w} = \varnothing.$$

Then from a continuity of the function $(g-h)\operatorname{sign}(f-g)$ on the set $X \backslash Z_{f-g} \supset M_{|f-g|+w}$ it follows that there exists an open set $N \supset M_{|f-g|+w}$ such that

$$[g(x)-h(x)]\operatorname{sign}[f(x)-g(x)] < 0$$

for all $x \in N$. Because a compact Hausdorff space X is a normal space, then there exists an open set U such that

$$M_{|f-g|+w} \subset U \qquad \text{and} \qquad \bar{U} \subset N.$$

Now from the construction of the set U and a compactness of $\bar{U}$ it follows that

$$\max\{[g(x)-h(x)]\operatorname{sign}[f(x)-g(x)]: x \in \bar{U}\} < 0 \tag{3}$$

and

$$\min\{|h(x)-g(x)|: x \in \bar{U}\} \\ \geqslant \min\{[h(x)-g(x)] \operatorname{sign}[f(x)-g(x)]: x \in \bar{U}\} > 0. \tag{4}$$

Additionally, the compactness of $\bar{U}$ and (3) implies that

$$\delta = \min\{|f(x)-g(x)|: x \in \bar{U}\} > 0.$$

Hence, it follows that there exists a sequence $\{g_i\}$ in G for the set D equal to $\bar{U}$ such that conditions (i) and (ii) in Definition 2 are satisfied. From (i) it follows that there exists an integer n such that $\|g-g_i\| < \delta$ for all $i \geqslant n$. Hence, from (ii), (3) and (4) we obtain

$$\min\{|g_i(x)-g(x)|: x \in \bar{U}\} > 0$$

and

$$\operatorname{sign}[f(x)-g(x)] = \operatorname{sign}[f(x)-g_i(x)] = \operatorname{sign}[g_i(x)-g(x)]$$

for every $i \geqslant n$ and $x \in \bar{U}$.

This implies that

$$|f(x)-g_i(x)| + w(x) \\ = ([f(x)-g(x)] - [g_i(x)-g(x)]) \operatorname{sing}[f(x)-g_i(x)] + w(x) \\ = |f(x)-g(x)| + w(x) - |g_i(x)-g(x)| < \||f-g|+w\|$$

for these i and x. If $\bar{U} = X$ then the proof is completed. Otherwise, for a compact set $V = X \backslash U$ we have $M_{|f-g|+w} \cap V = \varnothing$ and

$$\tau = \max\{|f(x)-g(x): x \in V\} < \||f-g|+w\|.$$

Now, (i) implies that there exists an integer m such that

$$\|g-g_i\| < \||f-g|+w\| - \tau$$

for all $i \geqslant m$.

Therefore we have

$$|f(x)-g_i(x)| + w(x) \leqslant |f(x)-g(x)| + w(x) + |g(x)-g_i(x)| \\ < \tau + \||f-g|+w\| - \tau = \||f-g|+w\|$$

for all $i \geqslant m$ and $x \in V$. From this inequality and that for $x \in \bar{U}$ it follows that the functions g_i for $i \geqslant \max(n, m)$ are better additive approximations to f than g. This completes the proof. ∎

Now, combining Theorems 1 and 2 we obtain the following result: If G has the weak betweenness property and $Z_{f-g} \cap M_{|f-g|+w} = \varnothing$ then the following theorem holds.

THEOREM 3. *A necessary and sufficient condition for an element $g \in G$ to be a best additive weight approximation in G to $f \in C(X)$ is that inequality* (1) *be satisfied for all $h \in G$.*

Note that Theorem 3 is reduced to the well-known Kolmogorov theorem in the case when G is a subspace of $C(X)$ and a weight function w is identically equal to zero on X.

DEFINITION 3. A family $\mathfrak{G}$ of subsets in $C(X)$ is K-maximal if a necessary and sufficient condition for Theorem 3 to hold for every $f \in C(X)$ is that $G \in \mathfrak{G}$.

THEOREM 4. *Let w be equal identically to zero. Then the family $\mathfrak{G}$ of subsets in $C(X)$ having the weak betweenness property is K-maximal.*

Proof. The sufficiency follows from Theorem 1 and 2. A simple proof of necessity was given in [9] (see also [7]). ∎

In particular, Theorem 4 implies that nonlinear approximating subsets in $C(X)$: asymptotically convex [4] and those having the betweenness property [1] also have the weak betweenness property, but the converse is not necessarily true. It is of interest to known whether Theorem 4 is true for $w \not\equiv 0$. This seems to be a very difficult problem, and consequently we leave it open.

3. ALTERNATION THEOREMS

In this section we shall give an alternation characterization of the best additive weight approximations based on the results of the previous section. In the next we shall assume that $X = [a, b]$ and use the following definitions taken from [5] (see also [2]).

DEFINITION 4. The subset G has property Z of degree n_g at $g \in G$ if for every $h \in G$ the function $(h - g)$ has at most $(n_g - 1)$ zeroes in $[a, b]$ or vanishes identically.

DEFINITION 5. G has property A of degree n_g at $g \in G$ if for given

(i) an integer m, $0 \leqslant m < n_g$,

(ii) a set $\{x_1, \ldots, x_m\}$ with $a = x_0 < x_1 < \cdots < x_m < x_{m+1} = b$,

(iii) ε with $0 < \varepsilon < \frac{1}{2}\min\{x_{j+1} - x_j : j = 0,\ldots, m\}$ and

(iv) a sign $\sigma \in \{-1, 1\}$

there exists $h \in G$ with $\|h - g\| < \varepsilon$ and

$$\operatorname{sign}[h(x) - g(x)] = \begin{cases} \sigma, & a \leqslant x \leqslant x_1 - \varepsilon, \\ (-1)^i \sigma, & x_i + \varepsilon \leqslant x \leqslant x_{i+1} - \varepsilon,\ i = 1,\ldots, m-1, \\ (-1)^m \sigma, & x_m + \varepsilon \leqslant x \leqslant b. \end{cases}$$

In the case $m = 0$, we require

$$\operatorname{sign}[h(x) - g(x)] = \sigma, \qquad a \leqslant x \leqslant b.$$

DEFINITION 6. The subset G has degree n_g at $g \in G$ if G has property Z and property A of degree n_g at g.

DEFINITION 7. The points a_i, $a \leqslant a_0 < \cdots < a_n \leqslant b$ are called alternation points of a function $f \in C([a, b])$ if

$$f(a_i) = (-1)^i f(a_0) \neq 0 \qquad \text{for } i = 1, 2,\ldots, n.$$

THEOREM 5. *Let G be a subset having property Z of degree n_g at $g \in G$. Then a sufficient condition for g to be a best additive weight approximation to f is that the set $M_{|f-g|+w}$ contain $(n_g + 1)$ alternation points of the function $f - g$ or $Z_{f-g} \cap M_{|f-g|+w} \neq \varnothing$.*

Proof. By Theorem 1 it is sufficient to prove the theorem when $M_{|f-g|+w}$ contain $(n_g + 1)$ alternation points of the function $f - g$ and $Z_{f-g} \cap M_{|f-g|+w} = \varnothing$. *Then inequality* (1) *may not be satisfied only if the function $g - h$ has at least n_g zeroes. Since G has property Z of degree n_g at $g \in G$ this is impossible.* ▮

Now we state two fundamental lemmas whose proofs were given in [6].

LEMMA 1. *A subset G of $C([a, b])$ having a degree at all $g \in G$ has the weak betweenness property.*

LEMMA 2. *Let g be an arbitrary fixed element of G and let $f \in C([a, b])$. Assume that G has a degree n_g at $g \in G$. Let D be a closed subset of $[a, b]$ such that $D \cap Z_{f-g} = \varnothing$. Then the following three conditions are equivalent:*

(i) *the set D contains at least $(n_g + 1)$ alternation points of the function $f - g$,*

(ii) *the inequality*

$$\max\{[g(x)-h(x)]\,\text{sign}[f(x)-g(x)] : x \in D\} \geqslant 0 \tag{5}$$

holds for all $h \in G$,

(iii) *the inequality*

$$\max\{[g(x)-h(x)]\,\text{sign}[f(x)-g(x)] : x \in D\} > 0 \tag{6}$$

is satisfied for all $h \in G$, *h being distinct from g.*

From Lemmas 1 and 2 and Theorems 2 and 5 we immediately obtain the following result: If G has a degree for all $h \in G$ and $D \cap Z_{f-g} = \varnothing$ where $D = M_{|f-g|+w}$, and if n_g denote a degree of G at g then the following theorem holds.

THEOREM 6. *The following four conditions are equivalent:*

(i) *the element g is a best additive weight approximation to f in G,*

(ii) *inequality* (5) *holds for all* $h \in G$,

(iii) *inequality* (6) *is satisfied for all* $h \in G$, $h \not\equiv g$,

(iv) *the set D contains at least* $(n_g + 1)$ *alternation points of the function* $f - g$.

DEFINITION 8. A family $\mathfrak{G}$ of subsets in $C([a, b])$ is A-maximal if a necessary and sufficient condition for Theorem 6 to hold for every $f \in C([a, b])$ is that $G \in \mathfrak{G}$.

From the above results and Lemmas 7.7 and 7.8 of [5, pp. 19–21] we obtain the following theorem.

THEOREM 7. *Let w be equal identically to zero. Then a family* $\mathfrak{G}$ *of subsets in* $C([a, b])$ *having a degree is A-maximal.*

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